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GLOBAL ANALYTIC REGULARITY FOR SUMS OF SQUARES OF VECTOR FIELDS

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Abstract. We consider a class of operators in the form of a sum of squares of vector fields with real analytic coefficients on the torus and we show that the zero order term may influence their global analytic hypoellipticity. Also we extend a result of Cordaro-Himonas.

1. Introduction and Results

Let Ω be an open set in $\mathbb{R}^N$, or more generally a real analytic manifold, and $\mathcal{A}(\Omega)$ be the set of real analytic functions in Ω. We shall consider operators of the form

$$P = -\sum_{j=1}^{\nu} X_j^2 + X_0 + a, \tag{1.1}$$

where $X_0, \ldots, X_\nu$, are real vector fields with coefficients in $\mathcal{A}(\Omega)$, and a is a complex valued function in $\mathcal{A}(\Omega)$. We shall discuss the analytic regularity of the solutions to the equation $Pu = f$, for a given function $f \in \mathcal{A}(\Omega)$. To be more precise and to state our results we shall need the following definitions. We recall that the operator P is said to be *analytic hypoelliptic* (*hypoelliptic*) in Ω if for any U open subset of Ω the conditions $u \in \mathcal{D}'(U)$ and $Pu \in \mathcal{A}(U)$ ($Pu \in C^\infty(U)$) imply that $u \in \mathcal{A}(U)$ ($u \in C^\infty(U)$). The operator P is said to be *globally* analytic hypoelliptic (hypoelliptic) in Ω if the conditions $u \in \mathcal{D}'(\Omega)$ and $Pu \in \mathcal{A}(\Omega)$ ($Pu \in C^\infty(\Omega)$) imply that $u \in \mathcal{A}(\Omega)$ ($u \in C^\infty(\Omega)$). Also, we recall that a point $x_0 \in \Omega$ is of *finite type* if the Lie algebra generated by the vector fields $X_0, \cdots, X_\nu$ spans the tangent space of Ω at x_0.

By the celebrated sum of squares theorem of Hörmander [Ho] the finite type condition is sufficient for the hypoellipticity of P in the more general case where P has C^∞ coefficients, while in the analytic category, which is our situation here, Derridj [D] proved that the finite type condition is also necessary for hypoellipticity. Baouendi and Goulaouic [BG] discovered that the finite type condition is not sufficient for the analytic hypoellipticity of P. They showed that if P is the operator in $\mathbb{R}^3$ defined by $P = (\partial_x)^2 + (x\partial_y)^2 + (\partial_t)^2$, then the equation $Pu = 0$ has a non-analytic solution near $x = 0$. After, several authors including Helffer

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[H], Pham The Lai-Robert [PR], Metivier [M1], Hanges-Himonas [HH1], [HH2], and Christ [Ch1], [Ch2] found different classes of operators satisfying the finite type condition and failing to be analytic hypoelliptic. In [CH], most of these classes of operators were proved to be globally analytic hypoelliptic on the torus. The purpose of this article is to extend Theorem 1.1 in [CH] for the case where lower order terms are present, and to show that if the vector field X_0 in (1.1) is complex, then the zero order term, a, may influence the global analytic hypoellipticity of P.

We start with the extension of a result in [CH].

Theorem 1.1. *Let P be an operator of the form (1.1) on the torus $\mathbb{T}^N = \mathbb{T}^m \times \mathbb{T}^n$, with variables $(x,t), x = (x_1,\dots,x_m),\ t = (t_1,\dots,t_n)$, and*

$$X_j = \sum_{k=1}^{n} a_{jk}(t)\frac{\partial}{\partial t_k} + \sum_{k=1}^{m} b_{j_k}(t)\frac{\partial}{\partial x_k},\ j = 0,\dots,\nu,$$

are real vector fields with coefficients in $\mathcal{A}(\mathbb{T}^n)$, and $a = a(x,t) \in \mathcal{A}(\mathbb{T}^{m+n})$ is complex-valued. If the following two conditions hold:

(i) *Every point of $\mathbb{T}^{m+n}$ is of finite type;*

(ii) *The vector fields $\sum_{k=1}^{n} a_{jk}(t)\frac{\partial}{\partial t_k},\ j = 1,\dots,\nu$, span $T_t(\mathbb{T}^n)$ for every $t \in \mathbb{T}^n$, then the operator P is globally analytic hypoelliptic in $\mathbb{T}^N$.*

Remark. A generalization of [CH] has been also obtained by Christ [Ch3] under the assumption of a certain symmetry condition, which does not hold here because of the dependence of a on x. A different generalization has been proved by Tartakoff [T3] under the restriction $\nu = n$, but with P in a more general form and assumed to satisfy a maximal estimate. However his method could be used for Theorem 1.1 too. Also, we mention the related work of Chen [C], Komatsu [Ko], Derridj-Tartakoff [DT], Metivier [M2], Sjöstrand [S], Tartakoff [T1], [T2], and Treves [T1]. Theorem 1.1 is only a partial result on the problem of global analytic hypoellipticity and there is no doubt that more general results are valid, although it is far from clear what is a necessary and sufficient condition for global analytic hypoellipticity.

Next, in the 2-dimensional torus we consider the case where in (1.1) X_0 is complex. While in the above theorem the zero order term did not play any role, we shall show that this is not the case in the following situation. In $\mathbb{T}^2$ let P be the operator defined by

$$P = -\bar{L}L + a,\ a \in \mathbb{C}, \tag{1.2}$$

where

$$L = \partial_t + ib(t)\partial_x,\ \text{ with } b \in \mathcal{A}(\mathbb{T}^1), \text{ and real-valued.} \tag{1.3}$$

Then we have the following results.

Theorem 1.2. *Let $t_0 \in \mathbb{T}^1$ be a zero of b of odd order. If $a \in \mathbb{C} - \{0\}$, then the operator P defined by (1.2) is analytic hypoelliptic near $\mathbb{T}^1 \times \{t_0\}$.*

Theorem 1.3. *Let P be as in (1.2). If all zeros of b are of odd order and if $a \in \mathbb{C} - \{0\}$, then P is globally analytic hypoelliptic. Conversely, if b has a zero of odd order and if $a = 0$, then P is not globally analytic hypoelliptic*

Such phenomena have been studied in the past for an operator on the Heisenberg group related to the Lewy operator by Stein [St], and Kwon [Kw].

2. Proof of Theorem 1.1

We start with a lemma about a global subelliptic estimate.

Lemma 2.1. *Let $X_j, j = 0, \dots, \nu$, be real C^∞ vector fields in $\mathbb{T}^N$ and $a \in C^\infty(\mathbb{T}^N)$. If all points of $\mathbb{T}^N$ are of finite type for $X_0, \dots, X_\nu$, then there exist $\varepsilon > 0$ and $C > 0$ such that*

$$\|u\|_\varepsilon \leq C\left(\|Pu\|_0 + \|u\|_{-1}\right), \; u \in C^\infty(\mathbb{T}^N), \tag{2.1}$$

where P is of the form (1.1).

Proof. Since the finite type condition holds at every point, there exists a local subelliptic estimate near each point (see [Ho], [K], [OR], [RS]) and this implies that the following property holds true: $u \in H^0(\mathbb{T}^N), \; Pu \in H^0(\mathbb{T}^N) \Longrightarrow u \in H^\epsilon(\mathbb{T}^N)$. Then by the closed graph theorem the following global estimate holds:

$$\|u\|_\varepsilon \leq C_1\left(\|Pu\|_0 + \|u\|_0\right), \; u \in C^\infty(\mathbb{T}^N), \tag{2.2}$$

for some $\varepsilon > 0$ and $C_1 > 0$.

By Lions' Lemma for any $\delta > 0$ there exists C_δ such that

$$\|u\|_0 \leq \delta\|u\|_\varepsilon + C_\delta\|u\|_{-1}, \; u \in C^\infty(\mathbb{T}^N). \tag{2.3}$$

Applying (2.3) in (2.2) and selecting δ appropriately small give (2.1). The proof of Lemma 2.1 is complete. □

Now let $u \in \mathcal{D}(\mathbb{T}^N)$ such that

$$Pu = f, \; \text{with } f \in \mathcal{A}(\mathbb{T}^N). \tag{2.4}$$

By Hörmander's theorem $u \in C^\infty(\mathbb{T}^N)$. To show that P is globally analytic hypoelliptic in $\mathbb{T}^N$ it suffices to show that

$$u \in \mathcal{A}(\mathbb{T}^N). \tag{2.5}$$

Since by our hypothesis P is elliptic in t, it suffices to show that there exists $B > 0$ such that

$$\|\partial_x^\alpha u\|_0 \leq B^{|\alpha|+1}\alpha!, \; \forall \alpha \in \mathbb{N}_0^m. \tag{2.6}$$

Since a and f are in $\mathcal{A}(\mathbb{T}^N)$, there exists $A > 0$ such that

$$\|\partial_x^\alpha a\|_\infty \leq A^{|\alpha|+1}\alpha!, \; \alpha \in \mathbb{N}_0^m, \tag{2.7}$$

and

$$\|\partial_x^\alpha f\|_0 \leq A^{|\alpha|+1}\alpha!, \; \alpha \in \mathbb{N}_0^m. \tag{2.8}$$

Since $\|u\|_0 \leq \|u\|_\varepsilon$, the basic inequality (2.1) implies the following weaker inequality:

$$\|u\|_0 \leq C(\|Pu\|_0 + \|u\|_{-1}), \; u \in C^\infty(\mathbb{T}^N), \tag{2.9}$$

which is what we need for proving (2.6). If we apply (2.9) with u replaced with $\partial_x^\alpha u$, then we obtain

$$\|\partial_x^\alpha u\|_0 \leq C(\|\partial_x^\alpha Pu\|_0 + \|[P, \partial_x^\alpha]u\|_0 + \|\partial_x^\alpha u\|_{-1}). \tag{2.10}$$

We have

$$\|\partial_x^\alpha u\|_{-1} \leq \|\partial_x^{\alpha - e_j} u\|_0, \tag{2.11}$$

where e_j is an element of the orthonormal basis of $\mathbb{R}^m$ such that the corresponding $\alpha_j \geq 1$. Also, by their form $X_j,\ j = 0, \dots, \nu$, commute with ∂_x^α and we have

$$[P, \partial_x^\alpha]u = a\partial_x^\alpha u - \partial_x^\alpha(au) = -\sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial_x^{\alpha-\beta} a \partial_x^\beta u.$$

Therefore

$$\|[P, \partial_x^\alpha]u\|_0 \leq \sum_{\beta<\alpha} \binom{\alpha}{\beta} \|\partial_x^{\alpha-\beta} a\|_\infty \|\partial_x^\beta u\|_0.$$

Then by using (2.6) and (2.7) we obtain

$$\|[P, \partial_x^\alpha]u\|_0 \leq \alpha! \sum_{\beta<\alpha} A^{|\alpha-\beta|+1} B^{|\beta|+1}. \tag{2.12}$$

By (2.8), (2.10), (2.11) and (2.12) we obtain

$$\|\partial_x^\alpha u\|_0 \leq C \left(A^{|\alpha|+1}\alpha! + \alpha! \sum_{\beta<\alpha} A^{|\alpha-\beta|} B^{|\beta|+1} + B^{|\alpha|}(\alpha - e_j)! \right). \tag{2.13}$$

We look for B of the form

$$B = MA, \text{ for some } M > 1, \tag{2.14}$$

such that (2.6) holds. By (2.13) it suffices to choose M such that for all $\alpha \in \mathbb{N}_0^m$ we have

$$C(A^{|\alpha|+1}\alpha! + \alpha! \sum_{\beta<\alpha} A^{|\alpha-\beta|+|\beta|+2} M^{|\beta|+1} + A^{|\alpha|} M^{|\alpha|}(\alpha - e_j)!) \leq A^{|\alpha|+1} M^{|\alpha|+1}\alpha!$$

By simplifying we obtain that the last inequality follows from

$$C \left(1 + AM \sum_{\beta<\alpha} M^{|\beta|} + \frac{1}{A} M^{|\alpha|} \right) \leq M^{|\alpha|+1}. \tag{2.15}$$

Since for $M > 1$ we have

$$\sum_{\beta<\alpha} M^{|\beta|} \leq \left[\left(\frac{M}{M-1} \right)^m - 1 \right] M^{|\alpha|}, \tag{2.16}$$

by (2.16) we see that for (2.15) to hold it suffices that

$$C \left(\frac{1}{M^{|\alpha|+1}} + A \left[\left(\frac{M}{M-1} \right)^m - 1 \right] + \frac{1}{AM} \right) \leq 1. \tag{2.17}$$

Since the left-hand side of (2.17) goes to zero as M goes to infinite, we conclude that there exist $M > 1$ such that (2.17) holds. And therefore (2.6) holds with $B = MA$. This completes the proof of Theorem 2.1.

3. Proof of Theorems 1.2 & 1.3

We start with a being a function of t; i.e. in $\mathbb{T}^2$ we consider the operator

$$P = -\bar{L}L + a,\ a = a(t) \in \mathcal{A}(\mathbb{T}^1), \tag{3.1}$$

where $L = \partial_t + ib(t)\partial_x$, with $b \in \mathcal{A}(\mathbb{T}^1)$, and real-valued. We shall work near a zero of $b(t)$, which for simplicity we will assume to be $t = 0$. Then we may assume that

$$b(t) = t^k g(t),\ g(t) \neq 0,\ -\delta \leq t \leq \delta,\ \text{some } \delta > 0. \tag{3.2}$$

If we expand P, we obtain

$$P = -\partial_t^2 - b^2(t)\partial_x^2 - ib'(t)\partial_x + a(t).$$

If $u \in C^\infty(\mathbb{T}_t, \mathcal{D}'(\mathbb{T}_x))$, then by taking Fourier transform with respect to x we obtain

$$\widehat{Pu}(\xi, t) = -\hat{u}_{tt}(\xi, t) + [\xi^2 b^2(t) + \xi b'(t) + a(t)]\hat{u}(\xi, t).$$

If we multiply by $\bar{\hat{u}}$ and integrate in $t \in (-\delta, \delta)$, then we obtain

$$\int_{-\delta}^{\delta} \widehat{Pu}(\xi, t)\bar{\hat{u}}(\xi, t)dt$$
$$= -\int_{-\delta}^{\delta} \hat{u}_{tt}(\xi, t)\bar{\hat{u}}(\xi, t)dt + \int_{-\delta}^{\delta}[\xi^2 b^2(t) + \xi b'(t) + a(t)]|\hat{u}(\xi, t)|^2 dt.$$

Then we integrate by parts and use the Cauchy-Schwarz inequality to obtain:

$$\begin{aligned} &\int_{-\delta}^{\delta} |\hat{u}_t(\xi, t)|^2 dt + \int_{-\delta}^{\delta}[\xi^2 b^2(t) + \xi b'(t)]|\hat{u}(\xi, t)|^2 dt \\ &\quad\leq \int_{-\delta}^{\delta}[\frac{1}{2} - a(t)]|\hat{u}(\xi, t)|^2 dt + \frac{1}{2}\int_{-\delta}^{\delta} |\widehat{Pu}(\xi, t)|^2 dt + |\bar{\hat{u}}(\xi, t)\hat{u}_t(\xi, t)\big|_{t=-\delta}^{\delta}|. \end{aligned} \tag{3.3}$$

Now let us assume that we have started with some $r > 0$. And δ above has been chosen to be in the interval $(0, r)$. If we assume that

$$Pu \in \mathcal{A}(\mathbb{T}^1 \times (-r, r)), \tag{3.4}$$

and we use the fact that the operator P is elliptic near $(\mathbb{T}^1 \times \{-\delta\}) \cup (\mathbb{T}^1 \times \{\delta\})$, then by (3.3) and (3.4) we obtain

$$\int_{-\delta}^{\delta} |\hat{u}_t(\xi, t)|^2 dt + \int_{-\delta}^{\delta}[\xi^2 b^2(t) + \xi b'(t)]|\hat{u}(\xi, t)|^2 dt \tag{3.5}$$

$$\leq \int_{-\delta}^{\delta}[\frac{1}{2} + \|a\|_\infty]|\hat{u}(\xi, t)|^2 dt + O(e^{-\varepsilon|\xi|}),$$

for some $\varepsilon > 0$. Next we shall absorb the term

$$(\frac{1}{2} + \|a\|_\infty)\int_{-\delta}^{\delta} |\hat{u}(\xi, t)|^2 dt$$

in the left-hand side of (3.5) by using the following (Poincaré inequality) argument. We write

$$\hat{u}(\xi, t) = \hat{u}(\xi, -\delta) + \int_{-\delta}^{t} \hat{u}_t(\xi, s)ds.$$

Then we obtain

$$|\hat{u}(\xi,t)|^2 \le 2c^2 e^{-2\varepsilon|\xi|} + 4\delta \int_{-\delta}^{\delta} |u_t(\xi,t)|^2 dt,$$

which implies that

$$\int_{-\delta}^{\delta} |\hat{u}(\xi,t)|^2 dt \le 4c^2 \delta e^{-2\varepsilon|\xi|} + 8\delta^2 \int_{-\delta}^{\delta} |u_t(\xi,t)|^2 dt. \tag{3.6}$$

If we choose δ such that (3.2) is true and furthermore $8\delta^2(\frac{1}{2} + \|a\|_\infty) < \frac{1}{2}$, then by using (3.6), relation (3.5) gives

$$\int_{-\delta}^{\delta} |u_t(\xi,t)|^2 dt + \int_{-\delta}^{\delta} [\xi^2 b^2(t) + \xi b'(t)] |\hat{u}(\xi,t)|^2 dt \lesssim e^{-\varepsilon|\xi|}. \tag{3.7}$$

Very similar to the above arguments applied to the operator $Q = -L\bar{L} + a$ give the inequality

$$\int_{-\delta}^{\delta} |v_t(\xi,t)|^2 dt + \int_{-\delta}^{\delta} [\xi^2 b^2(t) - \xi b'(t)] |\hat{v}(\xi,t)|^2 dt \lesssim e^{-\varepsilon|\xi|}, \tag{3.8}$$

for any $v \in C^\infty(\mathbb{T}_t, \mathcal{D}'(\mathbb{T}_x))$ with $Qv \in \mathcal{A}(\mathbb{T}^1 \times (-r,r))$. To summarize, we have the following lemma.

Lemma 3.1. *Let P be given by (3.1) with b as in (3.2), and $r > 0$ be a given number. If $\delta \in (0,r)$ is such that*

$$8\delta^2(\frac{1}{2} + \|a\|_\infty) < \frac{1}{2},$$

then the following hold:

1. *Any $u \in C^\infty(\mathbb{T}_t, \mathcal{D}'(\mathbb{T}_x))$ with $Pu \in \mathcal{A}(\mathbb{T}^1 \times (-r,r))$ satisfies inequality (3.7) for some $\epsilon > 0$.*

2. *Let $Q = -L\bar{L} + a$. Then any $v \in C^\infty(\mathbb{T}_t, \mathcal{D}'(\mathbb{T}_x))$ with $Qv \in \mathcal{A}(\mathbb{T}^1 \times (-r,r))$ satisfies inequality (3.8) for some $\epsilon > 0$.*

Now we assume that $b(t)$ has a **zero of odd order** at $t = 0$; without loss of generality we can assume

$$b'(t) \ge 0, \quad -\delta \le t \le \delta, \tag{3.9}$$

and we have the following proposition:

Proposition 3.2. *Let P be as in (3.1), b be as in (3.2) and (3.9), and $r > 0$ be a given number. If $\delta \in (0,r)$ is such that $8\delta^2(\frac{1}{2} + \|a\|_\infty) < \frac{1}{2}$, then the following hold:*

1. *For any solution $u \in C^\infty(\mathbb{T}_t, \mathcal{D}'(\mathbb{T}_x))$ to $Pu = f$, $f \in \mathcal{A}(\mathbb{T}^1 \times (-r,r))$ there exist constants $c > 0$ and $\varepsilon > 0$, which may depend on u, such that*

$$|\hat{u}(\xi,t)| \le c e^{-\varepsilon|\xi|}, \ \xi > 0, \ |t| \le \delta. \tag{3.10}$$

2. *For any solution $v \in C^\infty(\mathbb{T}_t, \mathcal{D}'(\mathbb{T}_x))$ to $Qv = f$, $f \in \mathcal{A}(\mathbb{T}^1 \times (-r,r))$ there exist constants $c > 0$ and $\varepsilon > 0$, which may depend on v, such that*

$$|\hat{v}(\xi,t)| \le c e^{-\varepsilon|\xi|}, \ \xi < 0, \ |t| \le \delta. \tag{3.11}$$

Remark. f may be assumed to satisfy the correct estimate only for $\xi > 0$ in (1), and $\xi < 0$ in (2).

Proof. Let $\xi > 0$. Then by (3.9) we obtain $\xi^2 b^2(t) + \xi b'(t) \geq 0$, and we can apply Lemma 4.1 in Cordaro-Himonas [CH] to show that

$$|\hat{u}(\xi,t)|^2 \lesssim \int_{-\delta}^{\delta} |\hat{u}_t(\xi,t)|^2 dt + \int_{-\delta}^{\delta} [\xi^2 b^2(t) + \xi b'(t)]|\hat{u}(\xi,t)|^2 dt. \tag{3.12}$$

Therefore by (3.7) and (3.12) we obtain (3.10).

If $\xi < 0$, then by (3.9) we obtain $\xi^2 b^2(t) - \xi b'(t) \geq 0$, and again we apply Lemma 4.1 in [CH] to obtain

$$|\hat{u}(\xi,t)|^2 \lesssim \int_{-\delta}^{\delta} |u_t(\xi,t)|^2 dt + \int_{-\delta}^{\delta} [\xi^2 b^2(t) - \xi b'(t)]|\hat{u}(\xi,t)|^2 dt. \tag{3.13}$$

By (3.8) and (3.13) we obtain (3.11). □

End of Proof of Theorem 1.2. Since by our hypothesis Pu is analytic, by Proposition 3.2 u satisfies the estimate (3.10). To complete the proof it suffices to show that u satisfies estimate (3.11) too. We have $L(-\bar{L}Lu + au) = Lf$. Since a is a constant, it commutes with L and we obtain $L(-\bar{L}L + a) = (-L\bar{L} + a)L$. Therefore we have that Lu satisfies the equation $(-L\bar{L} + a)(Lu) = Lf$. Now by applying the second part of Proposition 3.2 for $v = Lu$ we obtain that Lu satisfies estimate (3.11). If we solve the equation $-\bar{L}Lu + au = f$ for au, we obtain $au = \bar{L}(Lu) + f$. Since $a \neq 0$, we obtain

$$u = \frac{1}{a}(\bar{L}(Lu) + f).$$

Since both Lu and f satisfy estimate (3.11), the last relation implies that u satisfies the estimate (3.11) too. Since u satisfies both estimates (3.10) and (3.11), we obtain the inequality

$$|\hat{u}(\xi,t)| \lesssim e^{-\varepsilon|\xi|}, \ \xi \in \mathbb{R}. \tag{3.14}$$

Relation (3.14) together with standard arguments (see for example [CH]) implies that u is analytic near $\mathbb{T}^1 \times \{t_0\}$. This completes the proof of Theorem 1.2. □

To prove Theorem 1.3 we shall need the following result in Bergamasco [B].

Lemma 3.3. *Let L be as in (1.3) with $b \not\equiv 0$. Then L is globally analytic hypoelliptic in $\mathbb{T}^2$ if and only if the function $b(t)$ does not change sign in $\mathbb{T}^1$.*

Proof. If $b(t)$ does not change sign in $\mathbb{T}^1$, then condition (P) holds and by the work of Treves [Tr2] L is locally and therefore globally analytic hypoelliptic. If $b(t)$ does change sign, then by using the stationary phase method one can construct a non-analytic solution in $\mathbb{T}^2$ to $Lu = 0$, [B]. □

Proof of Theorem 1.3. If $a \neq 0$, then by Theorem 1.2 P is analytic hypoelliptic near $\mathbb{T}^1 \times \{t_0\}$, for each zero, t_0, of b. Therefore P is globally analytic hypoelliptic in $\mathbb{T}^2$. If $a = 0$, then $P = -\bar{L}L$. Since $b(t)$ changes sign, by Lemma 3.3 there exists a global non-analytic solution u to the equation $Lu = 0$ in $\mathbb{T}^2$. This implies $Pu = 0$, and therefore P is not globally analytic hypoelliptic in $\mathbb{T}^2$. This completes the proof of Theorem 1.3. □

4. Final remarks

1. If L is as in (1.3) and $a(t)$ is a real analytic function in $\mathbb{T}^1$, then Bergamasco can modify his arguments in [B] to show that Lemma 3.3 is also true for the operator $L + a$. Therefore the global analytic hypoellipticity of the operator $L + a$ is independent of a, while, by Theorem 1.3 this is not so for the operator $-\bar{L}L + a$.

2. A simple example of an operator L in $\mathbb{T}^2$ with $b \not\equiv 0$ and where the equation $Lu = 0$ has a non-analytic global solution is given by $L = \partial_t + i \sin t\ \partial_x$. The function $v = e^{-i(x+i(\cos t - 1))}$ is analytic in $\mathbb{T}^2$ and a solution to $Lv = 0$. Since $|v| = e^{\cos t - 1}$, we have that $|v| < 1$ for $t \neq 0$ and $|v| = 1$ for $t = 0$. If we let $u = \sqrt{1 - v}$, then u is a solution to $Lu = 0$, which is not in $C^1(\mathbb{T}^2)$. Here we used the branch of the square root $\sqrt{1-z}$ which is defined in $\mathbb{C} - [1, \infty)$.

References

[BG] M.S. Baouendi and C. Goulaouic, *Nonanalytic-hypoellipticity for some degenerate elliptic operators*, Bull. AMS **78** (1972), 483-486. MR **45:**5567

[B] A.P. Bergamasco, *Remarks about global analytic hypoellipticity*, Preprint (1995).

[CH] P. D. Cordaro and A. A. Himonas, *Global analytic hypoellipticity for a class of degenerate elliptic operators on the torus*, MRL **1** (1994), 501-510. MR **95j:**35048

[C] S.C. Chen, *Global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem on circular domains*, Invent. Math. **92** (1988), 173-185. MR **89h:**35240

[Ch1] M. Christ, *Certain sums of squares of vector fields fail to be analytic hypoelliptic*, Comm. in PDE **16(10)** (1991), 1695-1707. MR **92k:**35056

[Ch2] M. Christ, *A class of hypoelliptic PDE admitting non-analytic solutions*, Contemporary Mathematics, Madison Symposium in Complex analysis **137** (1992). MR **93j:**35048

[Ch3] M. Christ, *Global analytic hypoellipticity in the presence of symmetry*, MRL **1** (1994), 559-563. MR **95j:**35047

[D] M. Derridj, *Un probleme aux limites pour une classe d'operateurs du second ordre hypoelliptiques*, Ann. Inst. Fourier, Grenoble **21** (1971), 99-148. MR **58:**29139

[DT] M. Derridj and D. S. Tartakoff, *Global analyticity for $\Box_b$ on three dimensional CR manifolds*, Comm. in PDE **18(11)** (1993), 1847-1868. MR **94i:**32021

[HH1] N. Hanges and A. A. Himonas, *Singular solutions for sums of squares of vector fields*, Comm. in PDE 16 **8 & 9** (1991), 1503-1511. MR **92i:**35031

[HH2] N. Hanges and A.A. Himonas, *Analytic hypoellipticity for generalized Baouendi-Goulaouic operators*, Journal of Functional Analysis **125, No. 1** (1994), 309-325. MR **95j:**35049

[H] B. Helffer, *Conditions nécessaires d'hypoanalyticité pour des opérateurs invariants à gauche homogènes sur un groupe nilpotent gradué*, Journal of Differential Equations **44** (1982), 460-481. MR **84c:**35026

[Ho] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967/), 147-171. MR **36:**5526

[K] J. J. Kohn, *Pseudo-differential operators and hypoellipticity*, Proceedings of symposia in pure mathematics **XXIII** (1973), 61-69. MR **49:**3356

[Ko] G. Komatsu, *Global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem*, Tohoku Math. J. (2) **28** (1976), 145-156. MR **52:**14683

[Kw] K. H. Kwon, *Concatenations applied to analytic hypoellipticity of operators with double characteristics*, Trans. AMS **283** (1984), 753-763. MR **85j:**35045

[M1] G. Metivier, *Une class d'operateurs non hypoelliptiques analytiques*, Indiana Univ. Math J. **29** (1980), 823-860.

[M2] G. Metivier, *Analytic hypoellipticity for operators with multiple characteristics*, Comm. in PDE **1** (1981), 1-90. MR **82g:**35030

[OR] O. A. Oleinik and E. V. Radkevic, *Second order equations with nonnegative characteristic form*, AMS and Plenum Press (1973). MR **56:**16111

[PR] Pham The Lai and D. Robert, *Sur un problème aux valeurs propres non linéaire*, Israel J. of Math **36** (1980), 169-186. MR **83b:**35132

[RS] L.P. Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math **137** (1977), 247-320. MR **55:**9171

[S] J. Sjöstrand, *Analytic wavefront sets and operators with multiple characteristics*, Hokkaido Mathematical Journal **12** (1983), 392-433. MR **85e:**35022

[St] E. M. Stein, *An example on the Heisenberg group related to the Lewy operator*, Invent. Math. **69** (1982), 209-216. MR **84c:**35031

[T1] D. S. Tartakoff, *On the local real analyticity of solutions to* $\Box_b$ *and the* $\bar{\partial}$*-Neumann problem*, Acta Math **145** (1980), 117-204. MR **81k:**35033

[T2] D. S. Tartakoff, *On the global real analyticity of solutions to* $\Box_b$ *on compact manifolds*, Comm. in PDE **1(4)** (1976), 283-311. MR **53:**14552

[T3] D. S. Tartakoff, *Global (and local) analyticity for second order operators constructed from rigid vector fields on products of tori*, Trans. AMS **348** (1996), 2577-2583. MR **96i:**35018

[Tr1] F. Treves, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and applications to* $\bar{\partial}$*-Neumann problem*, Comm. in PDE **3** (1978), 475-642. MR **58:**11867

[Tr2] F. Treves, *Analytic hypoelliptic partial differential operators of principal type*, Comm. Pure Appl. Math. **24** (1971), 537-570. MR **45:**5569

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